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A 3 PARAMETER FAMILY OF SYMMETRIC GENERALIZED RIORDAN ARRAYS

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Abstract. The symmetric Pascal matrix and the symmetric array of Delannoy numbers are typical examples of symmetric generalized Riordan arrays. This paper proves that all symmetric generalized Riordan arrays are parametrized by three parameters and factorized as the product of generalized Pascal triangles, and also are totally positive.

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1. Generalized Riordan arrays

As intuitive methods of solving combinatorial problems, Riordan arrays are useful to build an understanding of many number patterns. Since Riordan arrays were first introduced in 1991 by Shapiro et al. [?], many studies and applications have been developed. Let $g(x) = \sum_{i=0}^{\infty} g_i x^i, f(x) = \sum_{i=0}^{\infty} f_i x^i$ be formal power series. A (proper) Riordan array $\mathcal{R} = [r_{nk}]$ is a triangular array each kth column of which is generated by $g(x)$ and $f(x)$ with $f_0 = 0, f_1 = 1$, that is, nth element of the kth column r_{nk} satisfies

$$r_{nk} = [x^n]g(x)f(x)^k,$$

where $[x^n]$ is the coefficient operator, and the first column and row are indexed by 0. For example, Pascal triangle $P = \left(\binom{i}{j}\right), (i, j = 0, 1, 2, \cdots)$ is a proper Riordan array.

The set of Riordan arrays has an algebraic structure by the multiplication law being given by

$$(g(x), f(x)) \cdot (h(x), l(x)) = (g((h \circ f)(x)), (l \circ f)(x)).$$

A (proper) Riordan array $(g(x), f(x))$ is a lower unit triangular array and is thus invertible. And the inverse array is also a Riordan array given by

$$(g(x), f(x))^{-1} = \left(1/g(f^{-1}(x)), f^{-1}(x)\right).$$

It follows that the set of Riordan arrays forms a group under matrix multiplication called the Riordan group.

A generalized Riordan array for column [?, ?] is an infinite array whose kth column is given by the coefficients in the expansion of the formal power series $g(x)f(x)^k$. We denote the generalized Riordan array for column by $(g(x), f(x))_C$. A generalized Riordan array for row is defined as an infinite

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array whose kth row is given by the coefficients in the expansion of the formal power series $g(x)f(x)^k$, and denoted by $(g(x), f(x))_R$. We note that the generalized Riordan arrays do not have the same generating function for columns and rows, in general. But well-known symmetric Pascal matrix $P_s = \left(\binom{i+j}{j}\right), (i, j = 0, 1, 2, \cdots)$ or the symmetric array of Delannoy numbers [?] $D = (d_{m,n}), d_{m,n} = \sum_{j=0}^{n} \binom{n}{j}\binom{n+j}{j}, (m, n = 0, 1, 2, \cdots)$ has the same generating function for columns and rows. In fact,

$$P_s = \left(\frac{1}{1-x}, \frac{1}{1-x}\right)_C = \left(\frac{1}{1-x}, \frac{1}{1-x}\right)_R,$$
$$D = \left(\frac{1}{1-x}, \frac{1+x}{1-x}\right)_C = \left(\frac{1}{1-x}, \frac{1+x}{1-x}\right)_R.$$

These generalized Riordan arrays with the same generating function for columns and rows are symmetric and simply denoted by $(g(x), f(x))$.

In general, a matrix multiplication of generalized Riordan arrays may not be well-defined. However, it is possible to multiply a generalized Riordan array $(g(x), f(x))_C$ and a generalized Riordan array of polynomial type $(p(x), P(x))_C$, where $p(x)$ and $P(x)$ are polynomials in x [?] as

$$(g(x), f(x)) \cdot (p(x), P(x)) = (g((p \circ f)(x)), (P \circ f)(x)).$$

This paper proves that all symmetric generalized Riordan arrays are parametrized by three parameters and factorized as the product of generalized Pascal triangles, and also are totally positive.

2. Symmetric Generalized Riordan arrays

Let $P(a, b)$ be generalized Pascal triangles defined by

$$P(a, b) = (p_{ij}), p_{ij} = a^{i-j} b^j \binom{i}{j}, (i, j = 0, 1, 2, \cdots).$$

Then $P(a, b)$ is a Riordan array $\left(\frac{1}{1-ax}, \frac{bx}{1-ax}\right)_C$ as generating functions for columns, and $(1, a + bx)_R$ as those for rows. These generalized Pascal triangles $P(a, b)$ are generalized Riordan arrays, but not symmetric, which means that the generating functions for columns and rows are not the same.

The following theorem gives the closed form of generating functions which determine symmetric generalized Riordan arrays.

Theorem 2.1. *A nonzero generalized Riordan array $M = (g, f)_R$ is symmetric if and only if $g = \dfrac{g_0}{1 - f_0 x}$ and $f = f_0 + \dfrac{f_1 x}{1 - f_0 x}$.*

Proof. Let $M = (g, f)_R$ be a nonzero symmetric generalized Riordan array. Then $M = (g, f)_R = (g, f)_C$ and $g_0 f_0^i = g_i$ for each i, which imply that $g = \frac{g_0}{1-f_0 x}$ is the ordinary generating function of the first row or column of M. The $(1, 2)$–element M_{12} and $(2, 1)$–element M_{21} of M satisfy

$$M_{12} = g_0 f_0^3 + g_0 f_0 f_1 + g_0 f_2,$$
$$M_{21} = g_0 f_0^3 + 2 g_0 f_0 f_1.$$

Since M is symmetric and nonzero, $f_2 = f_0 f_1$. Now to prove that by mathematical induction $f_{n+1} = f_0 f_n, \forall n \geq 1$, assume that $f_k = f_0 f_{k-1}, k \geq 2$. The $(1, k+1)$–element of M is

$$\begin{aligned} M_{1,k+1} &= \sum_{i=0}^{k+1} f_{k+1-i} g_i \\ &= \sum_{i=1}^{k} f_{k+1-i} g_i + f_0 g_{k+1} + f_{k+1} g_0 \\ &= \sum_{i=1}^{k} f_0^{k-i} f_1 g_0 f_0^i + f_0^{k+2} g_0 + f_{k+1} g_0 \\ &= k f_0^k f_1 g_0 + f_0^{k+2} g_0 + f_{k+1} g_0. \end{aligned}$$

And the $(k+1, 1)$–element of M is

$$M_{k+1,1} = [x]\left((g_0 + g_1 x)(f_0 + f_1 x)^{k+1}\right) = g_0 \binom{k+1}{k} f_0^k f_1 + g_1 f_0^{k+1}.$$

By the symmetricity of M,

$$k f_0^k f_1 g_0 + f_0^{k+2} g_0 + f_{k+1} g_0 = (k+1) g_0 f_0^k f_1 + g_0 f_0^{k+2}.$$

By the assumptions of $g_0 \neq 0$ and $f_k = f_0^{k-1} f_1$, $f_{k+1} = f_0^k f_1 = f_0 f_k$, which verifies that

$$f = f_0 + \frac{f_1 x}{1 - f_0 x}.$$

Conversely, assuming that $g = \dfrac{g_0}{1 - f_0 x}$ and $f = f_0 + \dfrac{f_1 x}{1 - f_0 x}$, then $g_i = g_0 f_0^i, f_n = f_0^{n-1} f_1 (n \geq 1)$. The $(1, n)$–element of M is

$$\begin{aligned} M_{1n} &= \sum_{i=0}^{n} f_{n-i} g_i \\ &= \sum_{i=0}^{n-1} f_0^{n-i-1} f_1 g_i + f_0 g_n \\ &= \sum_{i=0}^{n-1} f_0^{n-i-1} f_1 f_0^i g_0 + f_0^{n+1} g_0 \\ &= n f_0^{n-1} g_0 f_1 + f_0^{n+1} g_0, \end{aligned}$$

which is equal to M_{n1}, that is, M is a symmetric generalized Riordan array. □

By theorem **??**, the symmetric generalized Riordan array (g, f) is determined by three parameters g_0, f_0, f_1 only. For example, the symmetric Pascal triangle $P_s = \left(\dfrac{1}{1-x}, \dfrac{1}{1-x}\right)$ is the unique symmetric generalized Riordan array with initial values $g_0 = f_0 = f_1 = 1$.

To simplify notation, let $E(g_0, f_0, f_1)$ denote the symmetric generalized Riordan array $E = (g, f)$ determined by initial values g_0, f_0, f_1, and be called *Erin array with initial values* g_0, f_0, f_1.

Example 2.2. *(1) The Erin array* $E(1,0,1)$ *is the identity array* $(1,x)$*. And* $E(1,1,0)$ *is the array* $\left(\frac{1}{1-x},1\right)$ *that all elements are one.*

(2) The symmetric array of Delannoy numbers $D=\left(\frac{1}{1-x},\frac{1+x}{1-x}\right)$ *is the Erin array* $E(1,1,2)$*.*

Every Erin array has LU decomposition by generalized Pascal triangles as the following theorem.

Theorem 2.3. *(LU decomposition of Erin arrays)* $E(c,a,b)$ *is an Erin array if and only if there exist a lower triangular array* $L=c_lP(a,b_l)$ *and an upper triangular array* $U=c_uP(a,b_u)^T$ *for constants* $c_l,c_u,b_l,b_u\in\mathbb{R}$ *satisfying* $c=c_lc_u, b=b_lb_u$ *such that* $E=LU$*.*

Proof. Let $E(c,a,b)$ be an Erin array, and take constants $c_l,c_u,b_l,b_u\in\mathbb{R}$ satisfying $c=c_lc_u, b=b_lb_u$. Let $L=c_lP(a,b_l)$ and $U=c_uP(a,b_u)^T$ be lower and upper triangular arrays, respectively. Then

$$\begin{aligned}E(c,a,b)&=\left(\frac{c}{1-ax},a+\frac{bx}{1-ax}\right)\\&=\left(\frac{c_lc_u}{1-ax},a+\frac{b_lb_ux}{1-ax}\right)\\&=\left(\frac{c_l}{1-ax},\frac{b_lx}{1-ax}\right)_C\cdot(c_u,a+b_ux)_C\\&=LU.\end{aligned}$$

The converse also can be proved similarly. □

If $c_l,c_u,b_l,b_u\in\mathbb{R}$ are taken to be $c_l=c_u, b_l=b_u\in\mathbb{R}$, the following Cholesky decomposition theorem can be derived directly.

Theorem 2.4. *(Cholesky decomposition of Erin arrays)* $E=E(g_0,f_0,f_1)$ *is an Erin array with* $g_0,f_1\geq 0$ *if and only if there exist a lower triangular array* $L=cP(a,b)$ *for some nonzero constant* $c\in\mathbb{R}$ *such that* $E=LL^T$*.*

Example 2.5. *The symmetric array of Delannoy numbers* $D=\left(\frac{1}{1-x},\frac{1+x}{1-x}\right)$ *in Example* **??** *is the Erin array* $E(1,1,2)$*. By theorem* **??**

$$D=\left(\frac{1}{1-x},\frac{\sqrt{2}x}{1-x}\right)_C\cdot\left(1,1+\sqrt{2}x\right)_C.$$

A infinite matrix is said to be totally positive if its minors of all orders are nonnegative. For example, Pascal triangle and Cartan triangle are totally positive matrices[**?**, **?**]. By using theorem **??**, we have the following total positivity theorem.

Theorem 2.6. *All Erin arrays with nonnegative initial values are totally positive.*

Proof. Let $E(c,a,b)$ be an Erin array with $c,a,b \geq 0$. If $b=0$, the generalized Pascal triangles $P(a,b)$ are clearly totally positive. Assuming $b>0$, $E = cP(a,b_l)P(a,b_u)$ by theorem **??** where b_l, b_u are positive numbers such that $b_l b_u = b$. By classic Cauchy–Binet formula it suffices to show that $P(a,\alpha)$ is totally positive for each $\alpha > 0$. Note that the generalized Pascal triangles $P(a,\alpha) = (p_{ij})$ satisfy

$$p_{i+1,j} = \alpha p_{i,j-1} + a p_{i,j},$$

for $j \geq 1$, where $p_{0,0} = 1$. This recursion can be written as

$$P(a,\alpha) = \begin{pmatrix} 1/\alpha & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & a & \alpha & 0 & \cdots \\ 0 & a^2 & 2a\alpha & \alpha^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 & 0 & \cdots \\ a & \alpha & 0 & 0 & \cdots \\ 0 & a & \alpha & 0 & \cdots \\ 0 & 0 & a & \alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

or briefly,

$$P(a,\alpha) = \begin{pmatrix} 1/\alpha & 0 \\ 0 & P(a,\alpha) \end{pmatrix} T,$$

where T is a Toeplitz matrix with generating function $\alpha + az$. By Aissen–Schoenberg–Whitney Theorem T is totally positive. To show that $P(a,\alpha)$ is totally positive, it suffices to show that its leading principal submatrices are all totally positive. Let $P_n(a,\alpha)$ and T_n be denoted by the nth leading principal submatrices of $P(a,\alpha)$ and T respectively. Then

$$P_{n+1}(a,\alpha) = \begin{pmatrix} 1/\alpha & 0 \\ 0 & P_n(a,\alpha) \end{pmatrix} T_{n+1}.$$

By induction on n assuming that $P_n(a,\alpha)$ is totally positive, $\begin{pmatrix} 1/\alpha & 0 \\ 0 & P_n(a,\alpha) \end{pmatrix}$ is also totally positive. Since T_{n+1} is also totally positive, by classic Cauchy–Binet formula $P_{n+1}(a,\alpha)$ is totally positive. This implies that $P(a,\alpha)$ is totally positive. Thus for all nonnegative initial values c,a,b, Erin array $E(c,a,b)$ is totally positive. □

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